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Analysis of applications of Banach fixed point theorem

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In the context of normed space, Banach's fixed point theorem for mapping is studied in this paper. This idea is generalized in Banach's classical fixed-point theory. Fixed point theory explains many situations where maps provide great answers through an amazing combination of mathematical analysis. Picard-Lendell's theorem, Picard's theorem, implicit function theorem, and other results are created by other mathematicians later using this fixed-point theorem. We have come up with ideas that Banach's theorem can be used to easily deduce many well-known fixed-point theorems. Extending the Banach contraction principle to include metric space with modular spaces has been included in some recent research, the aim of study proves some properties of Banach space.

Keywords: Metric Space; Norm Space; Complete Norm Space; Banach.

1. INTRODUCTION

This study is inspired by previous research on Banach's fixed point theorem for finite mappings on graphs or metric spaces with fractional orders. Banach's fixed point theorem, associated with a complete standard space, is among the most important theorems. The development of Banach's fixed point theorem is motivated by the study of Banach's fixed point theorem and its applications. In his 1922 doctoral thesis, Polish mathematician Stefan Banach addressed Banach's fixed point theorem. Banach's fixed point theorem and Banach's contraction principle are crucial in this case for nonlinear analysis. It is a modification of Ekland's ε-variance principle ([1][2]), and is an essential tool in nonlinear analysis such as optimization. Differential equations, control theory, and variational inequalities. Subsequently, Banach's fixed point theorem has been extended and developed in several ways (see, e.g., [3-5] and associated references). We currently talk about Banach's fixed point theorem in modular space, rather than applying it in matrix space [6]. Finally, we demonstrate several important uses of Banach's fixed point theorem.

2. PRELIMINARIES

We will discuss Banach's fixed point theorem in metric spaces with full modular spaces and related topics. Metric space [7]: Let X be a non-empty set. Mapping K:X×X→R It is called a measure if ∀x,y,z∈X The following properties are satisfied
1) k(x,y)≥0.
2) k(x,y)=0 if and only if x=y.
3) k(x,y)=k(y,x) [Symmetry].
4) k(x,y)≤k(x,z)+k(z,y)[Triangle inequality].

When X is combined with the metric k, the result is referred to as a metric space. It is represented by k(X,k).
Example: In this simple but important case, the function provides the measure.

$$K(x,y)=\begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y \end{cases}$$

Definition 1

The convergence of the sequence and its bounds are determined by whether there is an x in the measure space (X,k) such that
limn→∞d(xn,x)= 0. $\lim_{n\to\infty} k(b_n , b) = 0$
Here x is called the limit of $\langle b_n \rangle$ and we write this$\lim_{n\to\infty} b_n = b$.
Definition 2

Cauchy sequence: Let (b_n) be a sequence in the metric space (X,k). If there is a positive integer N such that, for all ϵ>0, the sequence (b_n) is a Cauchy sequence.
k(b_p, b_q)<ϵ for all p,q>N

Definition 3

A gauge (X,d) is considered complete when all of its Cauchy sequences converge to one of its elements.

Definition 4

A fixed point of a mapping f:X→X is a point x∈X such that f(x)=x.

Definition 5

Let (X,d) be a metric space and consider contraction mapping in it. If for every m,n in M there exists a positive real number j<1, then the mapping T:M→N is referred to as a contraction on M.
k(Tn,Tm)≤jk(m,n)

Definition 6

A real function defined on M is ‖·‖:M → R defined on M such that for all α ∈ W, and for any m,n ∈M
1) ‖m‖≥0.
2) ‖m‖=0 if and only if m=0.

3) ‖λm‖=|λ|‖m‖.
4) ‖m+n‖≤‖m‖+‖n‖ (Triangle inequality).

The term "norm" refers to a measured on M defined by a norm on M and expressed as K(m,n)= ‖m−n‖; m,n∈M. (M,‖·‖) or just M is used to denote standard area.

Definition 7

The Banach space is a complete noem space. (Completeness refers to completeness within the scale specified by the standard.)

Remark: Every Banach space is a normed, but the converse, in general, is not true.

3. RESULTS AND CONCLUSIONS

Here we present a study of Banach's fixed point theorem and its applications to plot results, which are presented in standard spaces such as.

Theorem 1: Banach Contraction Theorem

Let P represent a contraction map over the entire metric space M. P has a single fixed point.
Proof:
Let $g_0 \in M$and let c_n define as:
g_0, g_1=Pg_0, g_2=Pg_1, g_3=Pg_2,···, g_n=Pg_{n-1}
Then, g_2=PPg_0=$p^2 g_0$

$$g_3 = p^2 p g_0 = p^3 g_0$$

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$$g_n = p^n g_0$$

Next, the image sequence of g_0 with P applied repeatedly. Now, we demonstrate that (g_n)

is a cauchy series. In the event where n>m, then
k(g_{m+1}, g_m)=k(Pg_m,Pg_{m-1})
⇒k(g_{m+1}, g_m)=≤vk(g_m, g_{m-1})=
⇒k(g_{m+1}, g_m)=≤vk(g_{m-1}, g_{m-2})=
⇒k(g_{m+1}, g_m)≤v2k(g_{m-1}, g_{m-2})=

Continuing in this manner until we reach m times,
k(g_{m+1}, g_m)≤$v^m k$ (g_1, g_0)
we obtain for n>m
k(g_{m+1}, g_n) ≤ k(g_m, g_{m+1})+k(g_{m+1}, g_{m+2})+···+k(g_{n-1}, g_n)
≤ $v^m k(g_1, g_0)$+ $v^{m+1} k(g_1, g_0)$+···+ $v^{n-1} k(g_1, g_0)$
=v^m(1+v+···+v_{n-m-1}) $k(g_1, g_0)$
= $v^m \frac{1-v_{n-m}}{1-V}$ $k(g_1, g_0)$
Since 0<V<1, So that the number 1− v_{n-m}<1
k(g_m, g_n)=$\frac{K^m}{1-K}$ k(g_1, g_0)

There exists a point g∈M since M is complete. so that g_n →g. We now demonstrate that the mapping P's limit x is a fixed point. Triangle inequality allows us to arrive at our definition.

K(g,Pg)≤k(g, g_n) + k(g_n,Pg)
⇒ K(g,Pg)≤ k(g, g_n) + Vk(g_{n-1},g)

When and only when g=q, we know that k(g,q)=0. This means that k(g, g_n) →0 and k(g, g_{n-1}) →0 since gn→g. As a result, Pg=g and k(g,Pg) = 0. g is a fixed point of P, as demonstrated by this. M is the only fixed point of P, as we have demonstrated. Assume that P's fixed point g_1 likewise exists. Consequently, P g_1= g_1.
k(g, g_n) = k(Pg,P g_n) ≤v k(g, g_n)
Since k<1, this implies that k(g, g_n)=0. Hence g= g_1

Theorem 2: Hahna-Banach (Normed Space)

Given a normal space M, let H be a bounded linear functional on a subspace N. In that case, h to is extended by a bounded linear functional H on M, which has the same norm.

$$\|H\|_g \le \|h\|_n$$

$\|H\|_g = \sup_{m\in M} |H(m)|, ||h(m)|| = \sup_{m\in M} |h(m)|$

Proof: H=0 and the extension h=0 follow if z={0}. Assume G≠{0}. We have for every m in G

$$|h(m)| = \|H\|_g \, ||m||$$

From theorem 3.1
|h(x)|≤k(x).
Where k(x)= $\|H\|_g \big||m|\big|$
Note that k(x) denotes on G
K(m+n)= $\|H\|_g \, ||m+n||$
By triangle inequality we get
K(m+n)= $\|H\|_g \, ||m|| + \|H\|_g \, ||n||$
K(m+n) = k(m)+k(n)
p(αm)= $\|H\|_g \, ||\alpha m||$
Hence by generalized Hahn-Banach theorem we can conclude that
|H(m)|≤k(m) = $\|H\|_g \, ||m||$
taking the supremum over all m∈G of norm 1
$\sup_{m\in M} |H(m)| \le \|h\|_g$
$|H(m)| \le \|h\|_g$
Since under an extension the norm cannot decrease then the theorem is proved.

Theorem 3

Let G is normed space then the following condition of continuous
1) (m,n)∈G×G → m+n∈G
2) (λ,m)∈W×G→λm∈W
3) (m,n)∈G×G→K(m,n)=‖m−n‖∈R

Proof: let (f,p)∈G×G be arbitrary point, and that its image is (f + p). This means that, given ε>0, ∃δ>0 S.t, whenever ||g−a‖<δ, ||(m+n)−(f+p)‖<ε ,‖n−p||<δ . Consider δ=1/2ε. Next, we have
‖(m+n)−(f+p)‖=‖(m−f)+(n−p)‖
⇒‖(m+n)−(f+p)‖≤‖m−f‖+‖n−p‖
⇒‖(m+n)−(f+p)‖<δ+δ
⇒‖(m+n)−(f+p)‖<ε/2+ε/2

⇒‖(m+n)−(f+p)‖<ε

Let f∈G and α∈D be arbitrary values. The mapping's continuousness at (α, f) will now be demonstrated. given ε>0, ∃δ>0 such that
‖λm−αf‖<ε
whenever ‖λ−α‖<δ and ‖x−a‖<δ we have the identity
λm−αf=(λ−α)(m−f)+λf+αm−αf−αf
⇒λm−αf=(λ−α)(m−f)+(λ−α)f+(m−f)α
⇒‖λm−αf‖=‖(λ−α)(m−f)+(λ−α)f+(m−f)α‖
⇒‖λm−αf‖≤‖(λ−α)(m−f)‖+‖(λ−α)f‖+‖(m−f)α‖
⇒‖λm−αf‖≤|λ−α| ‖m−f‖+|λ−α|‖f‖+|α|‖m−f‖
⇒‖λm−αf‖≤δδ+δ‖f‖+|α|δ<ε
⇒‖λm−αf‖<ε

Here, the distance of a metric space is function. The metric is continuous, which is inferred from metric spaces' characteristic.

Theorem 4: Banach Contraction Principle

There is a single fixed point g∈XG for each contraction mapping H defined on a Banach space G into itself. [14]
Proof: 1. exist of fixed point
Let g_0∈X and define the iterative sequence (g_n) by

$$g_0, g_1 = Hg_0, g_2 = Hg_1, \dots, g_n = Hg_{n-1}$$
$$g_2 = HHg_0 = H^2 g_0$$
$$g_3 = HH^2 g_0 = H^3 g_0$$

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$$g_n = H^n g_0$$

Let k=p+l,l=1,2,…then
$||g_{p+l} - g_s|| = ||H^{p+l}g_0 - H^s g_0|| = ||H\,(H^{p+l-1}g_0 - H^{s-1}g_0)|| <$ q‖ $H^{p+l-1}g_0 - H^{s-1}g_0||$
For n=1,2,3,…
$||H^n g_0$-$g_0||$=$||H^n g_0$-$H^{n-1}g_0$+$H^{n-1}g_0$-$H^{n-2}g_0$+$H^{n-2}g_0$+$H^n g_0$-$g_0||$
$||H^n g_0$-$g_0|| \le ||H^n g_0$-$H^{n-1}g_0||$+$||H^{n-1}g_0$-$H^{n-2}g_0||$+...+$||H^n g_0$-$g_0||$
$||H^n g_0$-$g_0|| \le ||H^{n-1}g_1$+$H^{n-1}g_0||$+$||H^{n-2}g_1$+$H^{n-2}g_0||$+…+$||g_1$-$g_0||$
$||H^n g_0$-$g_0|| \le l^{n-1}||g_1$-$g_0||$+$l^{n-2}||g_1$-$g_0||$+…+$||g_1$-$g_0||$
$||H^n g_0$-$g_0|| \le (l^{n-1} + l^{n-2} + \cdots .+1)\ ||g_1$-$g_0||$
$||H^n g_0$-$g_0|| \le \frac{1+l^n}{1-l}||g_1$-$g_0||$……(1)
Snice $1 + l^n < 0\ then\ by\ eq\ (1)$
$||H^n g_0$-$g_0|| \le \frac{1}{1-l}||g_1$-$g_0||$
n→ ∞ then

$$\lim_{n\to\infty} g_n = g$$

Snice H is continuous then

$$H_g = H(\lim_{n\to\infty} g_n) = \lim_{n\to\infty} g_{n+1} = g$$

Since limit g_{n+1}same limit g_n then g is fixed point
To prove the fixed point is unique we take two fixed point r ,g s.t

$H_g = g, H_r = r$ *then* $\left|\left|H_g - H_r\right|\right|$ ≤ l||g-r|| since H is continuous
||g-r|| ≤ l||g-r|| where l=1
||g-r||=0
Then g=r

Application Normed Space

Example 1: Let G=R be the Banach space of real numbers and ‖x‖=|x|, [a,b]⊂R, such that |h′(g)|≤k<1. Find the solution of the equation f(g)=g.
Proof:
Let g,r∈[v,s] and r<z<g. Then by Lagrange's mean value theorem we have

$$\frac{h(g) - h(r)}{g - r} = h'(\,z\,)$$

⇒h(g)−h®=(g−r)h′(z)
⇒|h(g)−h®|=|(g−r)h′(z)|
⇒|h(g)−h®|≤ l |g−r|

Therefore, by Banach contraction theorem exists a unique fixed point x∗∈[a,b] such that f(x∗)=x∗. Hence, x∗ is the solution of the equation f(x)=x

Example 2: Find the solution of the system with *n* unknowns

$$t_{11}g_1 + t_{12}g_2 + \cdots + t_{1n}g_n = q_1$$

$$t_{21}g_1 + t_{22}g_2 + \cdots + t_{2n}g_n = q_2$$

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$$t_{n1}g_1 + t_{n2}g_2 + \cdots + t_{nn}g_n = q_n$$

This system can be written as
g_1=(1-t_{11}) $g_1 - t_{12}g_2 - \cdots - t_{1n}g_n$+ q_1
g_2=$t_{21}g_1 + (1 - t_{22})g_2 + \cdots + t_{2n}g_n + q_2$
.
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.
g_n=$t_{n1}g_1 + t_{n2}g_2 + \cdots + (1 - t_{nn})g_n + q_n$
Let Hg=g then
||Hg-Hr||=||g-r||= sup |g-r|
||Hg-Hr||≤l sup|g-r|
This shows that *H* a contraction mapping of the Banach space into itself. Hence, by Banach contraction principle, there exists a unique fixed-point x.

4. CONCLUSIONS

There appear to be limitations to Banach's theory. Any continuous function that maps a unit interval to itself must have a logically fixed point. We expect that this study will be useful for fixed point theory and functional analysis related to modular spaces. We have obtained results that generalize known fixed point solutions in the context of Banach spaces to their standard spaces. Hence, each result predicted in this work will help us understand the best solution to the complex theory. In the future, we will talk about Banach spaces and how their modular spaces relate to physical problems.

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